

UCLA/93/TEP/3

February 1993

Universal T-matrix for Twisted Quantum $gl(N)$

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Abstract

The Universal T-matrix is the capstone of the structure that consists of a quantum group and its dual, and the central object from which spring the T-matrices (monodromies) of all the associated integrable models. A closed expression is obtained for the case of multiparameter (twisted) quantum $gl(N)$. The factorized nature of standard quantum groups, that allows the explicit expression for UT to be obtained with relative ease, extends to some nonstandard quantum groups, such as those based on $A_n^{(2)}$, and perhaps to all. The paper is mostly concerned with parameters in general position, but the extension to roots of unity is also explored, in the case of $gl(N)$. The structure of the dual is now radically different, and an interesting generalization of the q -exponential appears in the formulas for the Universal T- and R-matrices. The projection to quantum $sl(N)$ is simple and direct; this allows, in particular, to apply recent results concerning deformations of twisted $gl(N)$ to the semisimple quotient.

1. Introduction

1.1. PICTURES

Quantum groups may be studied in two different “pictures”, much as quantum mechanics has a Schroedinger picture and a Heisenberg picture. The Drinfeld picture is a quantum group from the point of view of integrable field theories, where the first examples were discovered. The development is due to Drinfeld [1], Jimbo [2] and many others. The Woronowicz-Manin picture appeared already in early work of Baxter [3] and Faddeev *et al.* [4], but the systematic study is due to Woronowicz [5] and Manin [6]. The term psuedo-group, proposed by Woronowicz in [5], deserves to be retained, in order that the terminology distinguish between this object and the quantum group in the sense of Drinfeld.

1.2. DUALITY

This paper is a contribution to the study of the duality between the two pictures. Much work has been done already [7], and a firm foundation that incorporates reflexivity and deformation theory has been discovered very recently [8], but the object that we regard as the capstone of the structure seems not to have been calculated except in some simple cases. It appears as the universal bi-character in Woronowicz and as a “canonical element” or “dual form” in other places. Here it will be called the Universal T-matrix, in recognition of the fact that the transition matrices of integrable models appear upon specialization, in passing from structure to representations. This is the main reason why it is useful for physics.

Let A be a vector space with a (countable) basis $\{|\alpha\rangle\}$. Let $\{<\beta|\}$ be the functions on A defined by

$$<\beta|\alpha\rangle = \delta_{\alpha\beta}. \quad (1.1)$$

To avoid or at least postpone the thorny question of an appropriate and precise definition of the dual space A^* , let us just say that these functions, and (at least) all finite linear combinations of them, are in A^* . The Universal T-matrix is the

unit operator in $End A$, namely

$$UT = \sum_{\alpha} |\alpha\rangle\langle\alpha|, \quad (1.2)$$

the “resolution of the identity.” It has been called the “canonical element of $A \otimes A^*$ ” and this expression can perhaps be justified once a convenient and rigorous definition of A^* has been agreed upon.

The notation was chosen in order to exhibit the connection between the object of main interest and familiar operations in quantum mechanics in Eq. (1.2). We now switch to a more convenient notation.

Let A be an associative algebra with basis $\{L_{\alpha}\}$, and $\{F_{\alpha}\}$ the functions defined by

$$F_{\alpha}(L_{\beta}) = \delta_{\alpha\beta}. \quad (1.3)$$

We shall suppose that the product $L_{\alpha}L_{\beta}$ is a finite linear combination of the basis elements; then a structure of coassociative coproduct is naturally induced on the linear span A^* of the functions F_{α} ; it is defined by

$$F_{\alpha} \mapsto \Delta F_{\alpha}, \quad \Delta F_{\alpha}(L_{\beta}, L_{\gamma}) = F_{\alpha}(L_{\beta}L_{\gamma}). \quad (1.4)$$

The expression for UT is

$$UT = T_{L,F} = \sum_{\alpha} L_{\alpha}F_{\alpha}, \quad (1.5)$$

and the relation (1.4) can be expressed as

$$T_{L,F}T_{L,F'} = \sum_{\alpha\beta} L_{\alpha}L_{\beta}F_{\alpha}F'_{\beta} = \sum_{\gamma} L_{\gamma}\Delta F_{\gamma}. \quad (1.6)$$

Here F, F' stands for two copies of A , $F_{\alpha}F'_{\beta}$ has the same meaning as $F_{\alpha} \otimes F_{\beta}$, and the result is that

$$T_{L,F}T_{L,F'} = T_{L,\Delta F}. \quad (1.7)$$

This is the first of the two structural relations that characterize the universal T-matrix.

Keep the notations as above, and suppose that A has, in addition, a coproduct that turns it into a bialgebra. Then A^* also becomes a bialgebra, with algebraic structure uniquely defined by

$$F_\alpha F_\beta(L_\gamma) = F_\alpha \otimes F_\beta(\Delta L_\gamma). \quad (1.8)$$

This too can be expressed in terms of UT :

$$T_{L,F} T_{L',F} = T_{\Delta L,F}. \quad (1.9)$$

This is the second structural relation satisfied by the Universal T-matrix; note that (1.7) and (1.9) are equivalent to (1.4) and (1.8).

Finally, suppose that A , as an algebra, is finitely generated by elements $\{\ell_i\}$ $i = 1, \dots, n$, and that each basis element L_α is an ordered monomial. We would like to answer the following.

Question. Under what condition is A^* finitely generated (as an algebra), and under what additional conditions is each element of the dual basis $\{F_\alpha\}$ a polynomial in the generators?

Though we do not know the answer in general, we can always use the relations (1.7) and (1.9) to determine the structure of A^* and thus answer the question on a case by case basis. Furthermore, when A^* turns out to be finitely generated we expect to obtain a useful expression for UT in terms of the two sets of generators.

1.3. EXAMPLES

Let A be the unital algebra freely generated by a single element x , with

$$\Delta x = x \otimes 1 + 1 \otimes x,$$

and take the basis $1, x, x^2, \dots$. Then

$$UT = \sum F_n x^n,$$

and the problem is to determine the dual structure. Eq. (1.9) gives

$$\sum F_m F_n x^m \otimes x^n = \sum F_k (x \otimes 1 + 1 \otimes x)^k$$

or

$$F_k F_1 = (k+1) F_{k+1} \tag{1.10}$$

with the unique solution ($p := F_1$)

$$F_k = p^k / k!$$

Hence A^* is the unital algebra freely generated by p , and

$$UT = e^{xp}. \tag{1.11}$$

Finally, (1.7) provides an easy evaluation of the coproduct:

$$\begin{aligned} e^{xp} e^{xp'} &= e^{x\Delta p} \\ \Rightarrow \Delta p &= p + p' = p \otimes 1 + 1 \otimes p. \end{aligned}$$

Next, let G be a Lie group and \mathcal{G} its Lie algebra with basis $\{\ell_i\}$ $i = 1, \dots, n$. Let A be the universal enveloping algebra of \mathcal{G} , with a basis $\{L_\alpha\}$ of ordered monomials, and Δ the unique compatible coproduct generated by

$$\Delta \ell_i = \ell_i \otimes 1 + 1 \otimes \ell_i. \tag{1.12}$$

[Compatible, that is, with the structure; $\ell_i \mapsto \Delta \ell_i$ generates a homomorphism.] Let $\{F_\alpha\}$ be the terms of functions on G at the identity defined by

$$L_\alpha F_\beta \big|_{Id} = \delta_{\alpha\beta}.$$

Then a calculation that follows precisely the pattern of the first example leads to a unique structure of bialgebra on the space A^* spanned by $\{F_\alpha\}$. This structure depends on the choice of the basis $\{L_\alpha\}$.

(i) If L_α is the symmetric product of $\{\ell_i\}$ $i \in \alpha$, then

$$F_\alpha = (1/|\alpha|!) \prod_{i \in \alpha} p_i.$$

The set α includes repetitions, $i \in \{1, \dots, n\}$, and $|\alpha|$ is the cardinality of α . The structure of A^* is Abelian,

$$UT = e^{p \cdot \ell}, \quad p \cdot \ell := \sum_{i=1}^n p_i \ell_i \quad (1.13)$$

and the coproduct $p_i \mapsto \Delta p_i$ is given by the Campbell-Hausdorff formula.

(ii) If L_α is an ordered polynomial, defined in terms of an ordering of the ℓ_i , for example $\ell_1 < \ell_2 < \dots < \ell_n$, then

$$F_\alpha = \prod_{i=1}^n (p_i)^{\alpha_i} / \alpha_i!,$$

where α_i is the incidence of i in α . The structure is Abelian,

$$UT = \prod_{i=1}^n e^{p_i \ell_i} \quad (1.14)$$

with the same ordering of factors, and the coproduct is expressed by another version of the Campbell-Hausdorff formula.

1.4. QUANTUM GROUPS

Our program is to obtain a formula for the Universal T-matrix of quantum groups analogous to (1.11) and (1.14). The problem of a quantum version of (1.13) seems to be more difficult [9].

For the quantum group $U_{q,q'}(g\ell_2)$ the formula

$$UT = e_a^{p_- x_-} e^{p_1 \rho_1} e^{p_2 \rho_2} e_{1/a}^{p_+ x_+}, \quad (1.15)$$

$$e_a^z := \sum z^n / [n!]_a, \quad a = 1/qq',$$

was given in [10]. The p_i generate the two-parameter (twisted) version of quantum $gl(2)$ in the sense of Drinfeld, with

$$[p_-, p_+] = (q - 1/q')(q^{p_1} q'^{p_2} - q'^{-p_1} q^{-p_2})$$

and the rest; the generators x_\pm, ρ_i of the dual generates a solvable Lie algebra with

$$[\rho_i, x_\pm] \propto x_\pm,$$

$$[\rho_1, \rho_2] = 0 = [x_+, x_-].$$

This is a special case of twisted, quantum $gl(N)$, treated in detail in this paper, with analogous results for all N .

The key to a simple generalization of (1.15) is to choose a preferred ordering of the generators. In this paper we take the Woronowicz-Manin pseudogroup (A) as our starting point, to end up (via the construction of UT) with the Drinfeld quantum group (A^*). The strategy is exactly the same as that followed in the above examples. The important question of ordering is related to a unique factorization of A , carried out in two steps. First

$$A = A_- \otimes_{A_0} A_+, \tag{1.16}$$

where A_\pm and A_0 are sub-bialgebras. Then

$$A_- \ni \tilde{X} = \prod_{i=1}^k \tilde{X}(i), \quad A_+ \ni \tilde{Y} = \prod_{i=1}^k \tilde{Y}(i) \tag{1.17}$$

where $\tilde{X}(i)$ and $\tilde{Y}(i)$ are nilpotent matrices arranged in a particular order, the choice of which is quite crucial.

The factorization, for the case of twisted, quantum $gl(N)$ is carried out with all details in Section 2, the Universal T-matrix is found in Section 3 and the structure

of the dual in Section 4. In this case A_0 is Abelian and the structure expressed by (1.16) is essentially the famous quantum double.

In Section 5 we show how the Universal R-matrix is obtained by a simple projection from UT , and in Section 6 we discuss the generalizations of (1.15) and (1.16) to

- (i) The standard quantization of any simple Lie algebra.
- (ii) Nonstandard quantum groups $(A_n^{(2)}, D_n^{(2)})$.
- (iii) Roots of unity.

Finally, in Section 7, we study the relation between twisted, quantum $gl(N)$ and $sl(N)$, and the question of rigidity of these quantum groups to further, essential deformations.

This paper does not include physical applications, but it may be worth repeating that the transfer matrices (monodromies) of solvable models (without spectral parameters) are obtained from UT by specialization to a representation of A . The choice of generators of A that is introduced via the factorization (1.16) and (1.17), gives a presentation of A as a deformed enveloping algebra, which greatly facilitates the construction of representations. This brings out the interesting problem of giving a sense to the Universal T-matrix for an affine quantum group (with spectral parameter). Applications without spectral parameter include ice-type models, knot theory and conformal field theory in two dimensions, but the inclusion of a spectral parameter seems to be required for more interesting physical applications. All this work on finite dimensional quantum groups must be regarded as preparation for an assault on the real problem.

1.5. TWISTED QUANTUM $gl(n)$

This quantum group was found independently by Reshetikhin [11], Schirmacher [12] and Sudbery [13]. In [11] it is offered as an example of “gauge transformations” on quantum groups based on simple Lie algebras. Since $gl(N)$ is not simple it is of some interest to study the relationship between twisted $gl(N)$ and twisted $sl(N)$.

The R-matrix of twisted, quantum $gl(N)$ is an element $R \in End(V \otimes V)$, $V = N$ -dimensional vector space, that in a particular basis is given by

$$R = \sum_i M_i^i \otimes M_i^i + \sum_{i < j} (q^{ji} M_j^j \otimes M_i^i + a q^{ij} M_i^i \otimes M_j^j + (1 - a) M_j^j \otimes M_i^i). \quad (1.18)$$

The parameters are q^{ij} with $q^{ij} q^{ji} = 1$ and a . The q 's are subject to change under “gauge transformation”; the parameter a is much more fundamental. The associated deformed permutation matrix P , often denoted \check{R} , is defined by

$$P_{ij}^{k\ell} = R_{ji}^{k\ell}; \quad (1.19)$$

it has two eigenvalues and satisfies the Hecke condition

$$(P - 1)(P + a) = 0. \quad (1.20)$$

Reference to “roots of unity” means that $a^K = 1$ for some (minimal) positive integer K . The observation that the Hecke parameter a , and not the q 's, is the relevant parameter, raises the question of identifying the important parameters in the case of quantum groups that do not satisfy the Hecke condition.

The matrix defined by (1.18) satisfies the Yang-Baxter relation

$$R_{12} R_{13} R_{23} = R_{23} R_{13} R_{12} \quad (1.21)$$

and the matrix P satisfies, in consequence, the braid relation

$$(braid)_{123} \equiv P_{12} P_{23} P_{12} - P_{23} P_{12} P_{23} = 0. \quad (1.22)$$

The next paragraph depends on (1.20) and (1.22), but not on the specific form of P that is implied by (1.18) and (1.19).

A quantum plane, in the sense of Manin, is an algebra generated by $\{x^i\}$ $i = 1, \dots, N$, with relations

$$xx(P-1)=0 \quad (x^i x^j P_{ij}^{k\ell} = x^k x^\ell). \quad (1.23)$$

A differential calculus D on this quantum plane, as defined most succinctly in [14], is the algebra generated by $\{x^i\}$ and $\{\theta^i\}$ $i = 1, \dots, N$, with additional relations

$$\theta\theta(P+a)=0, \quad (1.24)$$

$$a\theta x = x\theta P. \quad (1.25)$$

Definition. The pseudogroup $A(P)$ is the unital algebra generated by the matrix elements Z_i^j of an N -dimensional square matrix Z , with relations

$$[P, Z \otimes Z] = 0. \quad (1.26)$$

The following remarks also depend only on the validity of the Hecke relation and the braid relation.

Remark 1. One can look upon $A(P)$ as the algebra of quantum automorphisms of the differential calculus D , the action on D being generated by $x \mapsto xZ$, $\theta \mapsto \theta Z$ ($x^i \mapsto x^j Z_j^i$ etc.). The idea of “quantum automorphisms of the quantum plane” does not lead to $A(P)$.

Remark 2. The precise implication of the braid relation for the differential calculus D is contained in a theorem [15] that may be paraphrased as follows: The braid relation (1.22) is equivalent to the statement “ x commutes with (1.24) and θ commutes with (1.23).”

Remark 3. An implication of (1.26) is the existence of a unique compatible co-product on $A(P)$, such that

$$\Delta(z_i^j) = z_i^k \otimes z_k^j. \quad (1.27)$$

Henceforth we regard $A(P)$ as a bialgebra with this coproduct.

Remark 4. The statement (1.26) can be expressed as

$$ZZ(\mathcal{P} - 1) = 0, \quad (1.28)$$

where \mathcal{P} is the endomorphism defined by the matrix P acting on $\text{End } V \otimes V$. This matrix does not satisfy the Hecke condition (1.20), but instead

$$(\mathcal{P} - 1)(\mathcal{P} + a)(\mathcal{P} + a^{-1}) = 0. \quad (1.29)$$

thus $A(P)$ is a quantum plane with \mathcal{P} now taking the place of P and (1.29) replacing (1.20). Of course \mathcal{P} satisfies the braid relation.

It is natural to ask what is the algebra of quantum automorphisms of a quantum group, but the preceding remarks suggest something else. Regarding $A(P)$ as a quantum plane, one introduces a differential calculus \mathcal{D} on $A(P)$. The interesting object is the algebra of quantum automorphisms of this differential calculus.

Here we shall obtain (Section 3), in the case that A^* is the twisted, or multi-parameter, quantum group $U_{<q>,a}(gl_N)$, for generic a , the formula

$$UT = \prod_{\substack{i>j \\ m<n \\ k}} e_a^{X_i^j P_i^j} e^{\tau_k H_k} e_{1/a}^{Y_m^n Q_m^n} \quad (1.30)$$

The quantum group $A^* = U_{<q>,a}(gl_N)$ is generated by P_i^j (simple positive roots), Q_m^n (simple negative roots), and H_k (Cartan generators); the dual algebra $A(<q>, a)$ by X_i^j , τ_k and Y_m^n ; e_a^x is a deformed exponential.

The structure of bialgebra on $A(A^*)$ determines the structure of bialgebra on the dual, so that UT can be calculated along with the structure of $A^*(A)$. The strategy followed in this paper was to begin from the structure of the pseudo-group $A = A(<q>, a)$ (Woronowicz-Manin picture), and use the structural properties of

the dual form to determine, first UT and then the structure of the quantum group $A^* = U_{<q>,a}(gl_N)$ (Drinfeld picture). The structural relations are, once more

$$T(x, l) T(x, l') = T(x, \Delta(l)), \quad (1.31)$$

$$T(x, l) T(x', l) = T(\Delta(x), l). \quad (1.32)$$

Here x and x' refers to two copies of A and l, l' to two copies of A^* ; $\Delta(x)$ is the coproduct of A and $\Delta(l)$ is the coproduct of A^* . The first relation determines the algebraic structure of A^* and the second one gives the coproduct (Section 4).

There are interesting homomorphisms into a quantum group from its dual, that allow us to obtain the Universal R-matrix from the expression for the Universal T-matrix. If Φ is such a homomorphism, then

$$UR = (id \otimes \Phi)UT; \quad (1.33)$$

Φ operates on the generators of $A(<q>, a)$. (Section 6)

The similarity of (1.3) to (1.2) seems to suggest that the formula may be useful in connection with Fourier transforms on quantum groups.

2. Factorization

Let R be the Yang-Baxter matrix (1.18), in the fundamental representation of multiparameter [12, 13] (twisted [11]) quantum $gl(N)$, V an N -dimensional vector space, $P \in End(V \otimes V)$ defined by

$$P_{ji}^{kl} = R_{ij}^{kl}. \quad (2.1)$$

Then P satisfies the braid relation,

$$(\text{braid})_{123} \equiv P_{12}P_{23}P_{12} - P_{23}P_{12}P_{23} = 0, \quad (2.2)$$

and the Hecke condition (with a in the field)

$$(P - 1)(P + a) = 0. \quad (2.3)$$

Let F be the algebra of formal power series finitely generated by (z_i^j) , $i, j = 1, \dots, N$, $(z_i^i)^{-1}$, $i = 1, \dots, N$, and the unit, with relations

$$z_i^i (z_i^i)^{-1} = (z_i^i)^{-1} z_i^i = 1, \quad i = 1, \dots, N.$$

The quantum algebra (pseudogroup [5]) $A = A(< q >, a)$ is the quotient of F by the ideal generated by the relations

$$[P, Z \otimes Z] = 0, \quad Z \equiv \text{matrix } (z_i^j). \quad (2.4)$$

There is a unique algebra homomorphism $\Delta : A \rightarrow A \otimes A$, such that $\Delta(Z) = Z \otimes T$; that is,

$$\Delta(z_i^j) = \sum_k z_i^k \otimes z_k^j, \quad (2.5)$$

which gives A a structure of bialgebra.

There is more than one sense in which $A(< q >, a)$ is dual to the quantum group $U_{< q >, a}(gl_N)$, a deformation of the enveloping algebra $U(gl_N)$ of $gl(N)$. Recall that (the differential of) the classical r -matrix defines a Lie structure on the dual space $gl(N)^*$, turning $\{gl(N), gl(N)^*\}$ into a Lie bialgebra dual pair. Here we shall relate $A(< q >, a)$ to a deformation $U_{< q >, a}(gl_N^*)$ of the enveloping algebra of $gl(N)^*$. Then we show that there is a natural duality between the two deformed enveloping (bi-)algebras.

The first step is to justify the following factorization:

$$z_i^j = \sum_k X_i^k z_k Y_k^j, \quad (2.6)$$

$$X_i^j = \begin{cases} 1, & i = j, \\ 0, & i < j, \end{cases} \quad Y_i^j = \begin{cases} 1, & i = j, \\ 0, & i > j, \end{cases} . \quad (2.7)$$

Let F' be the algebra generated by X_i^j , z_k , $(z_k)^{-1}$ and Y_i^j (with unit), with relations $z_k(z_k)^{-1} = (z_k)^{-1}z_k = 1$ and (2.7). The formula (2.6) is invertible and allows the identification of F' with F . The relations (2.4) make sense in F' and the associated quotient is an alternative presentation of A . We shall obtain the alternative expressions for the relations and show, in particular, that the X 's commute with the Y 's.

Proposition. Let A_- , A_+ and A_0 be quotients of A by the ideals generated by

$$I_- = \{z_i^j, i < j\}, \quad I_+ = \{z_i^j, i > j\}, \quad I_0 = \{z_i^j, i \neq j\}, \quad (2.8)$$

then

$$A = A_- \otimes_{A_0} A_+. \quad (2.9)$$

Proof. The crucial ingredient is the fact that the sets $I_{\pm,0}$ actually generate ideals of A . This is an easy consequence of the relations, but there is a deeper reason for it. It is known that A has representations π and π' given by

$$(\pi(z_i^k))_j^l = R_{ij}^{kl}, \quad (\pi'(z_i^k))_j^l = (R^{-1})_{ji}^{lk}, \quad (2.10)$$

reflecting the existence of two homomorphisms from $gl(N)^*$ to $gl(N)$. The kernels of these two isomorphisms include the sets I_- and I_+ , respectively. Now let (\tilde{X}_i^j) and (\tilde{Y}_i^j) be the images of (z_i^j) under the projections $A \rightarrow A_-$ and $A \rightarrow A_+$, then $\tilde{X}_i^j = 0$ for $i < j$ and $\tilde{Y}_i^j = 0$ for $i > j$, and define

$$\tilde{z}_i^j \equiv \sum_k \tilde{X}_i^k \otimes \tilde{Y}_k^j. \quad (2.11)$$

The relations among the \tilde{X} 's and among the \tilde{Y} 's are exactly the same as the relations among the z 's and, because Δ is a homomorphism, the same relations are also obeyed by the \tilde{z} 's. The composite mapping given by the two projections of A on

A_+ and on A_- followed by (2.11) is one-one, so we can identify \tilde{z}_i^j with z_i^j . Now set

$$\tilde{X}_i^j = X_i^j x_j, \quad i \geq j, \quad \tilde{Y}_i^j = y_i Y_i^j, \quad i \leq j, \quad X_i^i = Y_i^i = 1, \quad (2.12)$$

and identify X_i^j with $X_i^j \otimes 1$, Y_i^j with $1 \otimes Y_i^j$, so that

$$z_i^j = X_i^k z_k Y_k^j, \quad z_k \equiv x_k \otimes y_k. \quad (2.13)$$

This is the desired formula (2.6). It is now evident that the X 's commute with the Y 's, (2.13) is precisely what we mean (2.9), and the proposition is proved.

As we said, the relations among the \tilde{X} 's and among the \tilde{Y} 's are exactly the same as among the z 's, and one easily derives the following relations among the generators of the decomposition (2.13). First,

$$z_i z_j = z_j z_i,$$

$$z_k X_i^j = C_{kij} X_i^j x_k, \quad z_k Y_i^j = C'_{kij} Y_i^j z_k. \quad (2.14)$$

The coefficients C_{kij} and C'_{kij} are given in the Appendix, Eq. (A.3-3'). Next, define "simple generators"

$$X_i \equiv X_{i+1}^i, \quad Y_i = Y_i^{i+1}, \quad i = 1, \dots, N,$$

then there are quommutation relations and Serre relations

$$[X_i, X_j]_{k_{ij}} = 0, \quad [Y_j, Y_i]_{k_{ij}} = 0, \quad |i - j| > 1, \quad (2.15)$$

$$[X_{i+1}, X_i]_{k_i} = (1 - 1/a) X_{i+1}^{i-1}, \quad (2.16)$$

$$[Y_i, Y_{i+1}]_{k_i} = -(1 - 1/a) Y_{i+1}^{i-1}, \quad (2.17)$$

$$[X_i, [X_{i+1}, X_i]_{k_i}]_{r_i} = 0 = [X_{i+1}, [X_{i+1}, X_i]_{k_i}]_{s_i}, \quad (2.18)$$

$$[[Y_i, Y_{i+1}]_{k_i}, Y_i]_{r_i} = 0 = [[Y_i, Y_{i+1}]_{k_i}, Y_{i+1}]_{s_i}. \quad (2.19)$$

Here $[A, B]_k \equiv AB - kBA$. The coefficients k_{ij} and k_i are in (A.4).

This alternative presentation of A , in terms of X_i^j, Y_i^j, z_k and $(z_k)^{-1}$, is very convenient for our purpose. In particular, the construction of the universal T -matrix for A reduces to the same problem for A_{\pm} . As an ultimate refinement, we introduce elements ρ_k of A_- , σ_k of A_+ and τ_k of A by setting

$$x_k = e^{\rho_k}, \quad y_k = e^{\sigma_k}, \quad z_k = e^{\tau_k}, \quad (2.20)$$

and adopt ρ_k, σ_k, τ_k as generators instead of x_k, y_k, z_k . By abuse of notation we still use the same names, A_{\pm} and A . As is very well known, and obvious in the sequel, these algebras must be completed with some infinite series, including the series (2.20). Care must be taken to extend far enough to get closure under the algebraic operations, without making the algebras too large to be manageable. For such questions we refer to the papers [15] and [16].

In this new presentation, in which (τ_k) replace (z_k^k) and $(z_k^k)^{-1}$ as generators, A becomes a deformation $U_{<q>,a}(gl_N^*)$ of the enveloping algebra of the Lie algebra $gl(N)^*$ (with the Lie structure determined by the classical r -matrix). The Serre presentation of $U_{<q>,a}(gl_N^*)$ is easily obtained from (2.14-19).

3. The Universal T-Matrix

The construction has already been explained elsewhere [10]. For A_- we take the basis

$$X^{[a][\alpha]} \equiv \prod_{\substack{i>j \\ k}} (X_i^j)^{a_{ij}} (\rho_k)^{\alpha_k}, \quad e^{\rho_k} \equiv x_k. \quad (3.1)$$

To facilitate the manipulations that follow, it is crucial to adopt a good ordering of the factors in the definition of the basis elements; the good rule is that X_i^j precedes X_l^k if $j < k$ or $j = k, i < l$.

A rigorous definition of the dual is based on the natural basis that is provided by the functions defined by

$$P_{[a][\alpha]}(X^{[b][\beta]}) = \begin{cases} 1, & \text{if } [a][\alpha] = [b][\beta], \\ 0, & \text{otherwise.} \end{cases}$$

For details concerning completion of the dual algebra in terms of entire functions on A we refer to the papers [15] by and [17].

If $\{P_{[a][\alpha]}\}$ is the dual basis, then the universal T -matrix for A_- is

$$T^- = \sum_{[a][\alpha]} X^{[a][\alpha]} P_{[a][\alpha]}. \quad (3.2)$$

The important structural properties of this operator were discussed in the Introduction; Eq. (1.32) for T^- reads

$$\begin{aligned} \sum \left(\prod_{\substack{i>j \\ k}} (X_i^j)^{a_{ij}} (\rho_k)^{\alpha_k} \right) \left(\prod_{\substack{i>j \\ k}} (X_i'^j)^{b_{ij}} (\rho'_k)^{\beta_k} \right) P_{[a][\alpha]} P_{[b][\beta]} \\ = \prod_{\substack{i>j \\ k}} (\Delta X_i^j)^{c_{ij}} (\Delta \rho_k)^{\gamma_k} P_{[c][\gamma]}. \end{aligned} \quad (3.3)$$

We have abandoned the cumbersome notation with \otimes , writing X for $X \otimes 1$ and X' for $1 \otimes X$ from now on. Comparing coefficients of both sides one gets the relations satisfied by the dual basis. As an example, consider the coefficients of $(X_i^j)^{n-1} X_i'^j$ for a fixed pair (i, j) . On the left one has (dots standing for zeros) $P_{[...,n-1,...][0]} P_{[...,1,...][0]}$; and on the other side the only contributing term is

$$(\Delta X_i^j)^n P_{[...,n,...][0]}. \quad (a_i^j = n, \quad \text{all others zero}).$$

[This at first sight innocent statement is valid because of the particular order chosen.] We need to know that the coefficient of $(X_i^j)^{n-1} X_i'^j$ in $(\Delta X_i^j)^n$ is

$$[n]_a \equiv (a^n - 1)/(a - 1), \quad (3.4)$$

to get a simple recursion relation for $P_{[\dots, n, \dots][0]}$, with the solution

$$P_{[\dots, n, \dots][0]} = (P_i^j)^n / [n!]_a, \quad (3.5)$$

provided a is not a root of 1.

The result is that the dual of A_- is generated by

$$(P_i^j), \quad i > j, \quad \text{and} \quad H_k = P_{[0][\dots, 1, \dots]}, \quad 1 \text{ in } k\text{'th place},$$

and that

$$T^- = \prod_{\substack{i > j \\ k}} e_a^{X_i^j P_i^j} e^{\rho_k H_k}.$$

A similar calculation gives T^+ , and the product $T^- T^+$, with the identification $\rho_k + \sigma_k = \tau_k$ is the Universal T -matrix for $A(< q >, a)$:

$$UT = \prod_{\substack{i > j \\ m < n \\ k}} e_a^{X_i^j P_i^j} e^{\tau_k H_k} e_{1/a}^{Y_m^n Q_m^n}. \quad (3.6)$$

The structure of UT reflects that of (2.6) and (2.9).

4. Relations and Coproduct of Twisted, Quantum $gl(N)$

Such relations are of course known [18], but we want to show that they drop out of our formula for the universal T -matrix, and that the generators P_i^j , H_k and Q_i^j are precisely the generators of a conventional presentation. Actually, we have found earlier derivations difficult to understand, and we hope that the one given here may be an improvement.

First, in

$$T_{x,p}^- T_{x',p}^- = T_{\Delta x,p}^- \quad (4.1)$$

we compare the coefficients of (properly ordered) elements of the basis of A . We first notice that terms linear in X_i^j and ρ'_k , for a fixed triple (i, j, k) , occur with the same coefficients on both sides, as a direct result of our construction. But terms involving $\rho_k X_i'^j$ are in the wrong order; the coefficient on the left is $H_k P_i^j$, and on the right side it is found by examination of the term

$$P_i^j \Delta(X_i^j) e^{H_k \Delta(\rho_k)} = P_i^j \Delta(X_i^j) (1 + H_k \Delta(\rho_k) + \dots). \quad (4.2)$$

Applying (A.6-10) one sees that the relevant part of $\Delta(X_i^j)$ is contained in

$$(x_i/x_j) X_i'^j = (1 + \rho_i - \rho_j + \dots) X_i'^j;$$

the coefficient we are looking for is thus $P_i^j (H_k + \delta_{ik} - \delta_{jk})$, and

$$[H_k, P_i^j] = (\delta_{ki} - \delta_{kj}) P_i^j. \quad (4.3)$$

By these means one easily recovers the complete Serre presentation of the positive Borel subalgebra of the quantum group $U_{<q>,a}(gl_N)$. The “simple” generators are $P_i = P_{i+1}^i$, $i = 1, \dots, N$, and the remaining relations are

$$\begin{aligned} [P_j, P_i]_{k_{ij}} &= 0, \quad \text{if } |i - j| > 1, \\ [[P_i, P_{i+1}]_{k_i}, P_i]_{r_i} &= [[P_i, P_{i+1}]_{k_i}, P_{i+1}]_{s_i} = 0. \end{aligned} \quad (4.4)$$

Next, we get the commutator between the simple generators of $U_{<q>,a}(gl_N)$,

$$P_i \equiv P_{i+1}^i \quad \text{and} \quad Q_i \equiv Q_i^{i+1}, \quad (4.5)$$

by comparing coefficients of $X_{i+1}^i Y_i^{i+1}$ on both sides of $T_{x,p} T_{x',p} = T_{\Delta x,p}$. For this we need to know parts of the expression for $\Delta(\rho_k)$. This is found in the Appendix; the result is that, for $i = 1, 2, \dots, N - 1$,

$$[P_i, Q_j] = \delta_i^j \frac{a}{1-a} (q^{i,i+1})^{1-H_{i+1}-H_i} (a^{-H_{i+1}} - a^{-H_i}) C_i. \quad (4.6)$$

The relations

$$\begin{aligned}
[H_k, Q_i^j] &= (\delta_{ki} - \delta_{kj}) Q_i^j \\
[Q_i, Q_j]_{k_{ij}} &= 0, \quad \text{if } |i - j| = 1, \\
[Q_i, [Q_{i+1}, Q_i]_{k_i}]_{s_i} &= [Q_{i+1}, [Q_{i+1}, Q_i]_{k_i}]_{s_i} = 0,
\end{aligned} \tag{4.7}$$

complete the structure. The dependence on $q^{i,i+1}$ can be removed by a slight redefinition of the simple generators to make the structure reduce to that of standard quantum $gl(N)$, as found in [18], but that would mess up the expression (3.6).

The coproduct is obtained from the other structural formula, Eq. (1.4),

$$\prod_{\substack{i>j \\ m<n \\ k}} e_a^{X_i^j P_i^j} e^{\rho_k H_k} e^{Y_m^n Q_m^n} \prod_{\substack{i>j \\ m<n \\ k}} e_a^{X_i^j P_i^j} e^{\rho_k H'_k} e^{Y_m^n Q_m^n} = \prod_{\substack{i>j \\ m<n \\ k}} e_a^{X_i^j \Delta P_i^j} e^{\rho_k \Delta H_k} e^{Y_m^n \Delta Q_m^n}, \tag{4.8}$$

it is the homomorphism generated by

$$\begin{aligned}
\Delta(H_k) &= H_k \otimes 1 + 1 \otimes H_k, \\
\Delta(P_i) &= P_i \otimes 1 + A_i \otimes P_i, \quad \Delta(Q_i) = Q_i \otimes B_i + 1 \otimes Q_i. \\
A_i &= C_i(q^{i+1,i})^{H_i+H_{i+1}} a^{-H_i}, \quad B_i = C_i(q^{i+1,i})^{H'_i+H'_{i+1}} a^{-H'_{i+1}}, \\
C_i &= \prod_{k \neq i, i+1} (q^{i+1,k} q^{ki})^{H_k}.
\end{aligned} \tag{4.9}$$

5. Applications of the Universal T-Matrix

We noted the existence of two representations of the quantum algebra $A(< q >, a)$ in $gl(N)$. If, in the formula (3.6), one takes the generators P, H and Q in the fundamental representation, then one recovers the original N -dimensional matrix $Z = (z_i^j)$, and if one then takes z_i^j in the representation (2.10), then one gets the original R -matrix in the fundamental representation. Our results concerning the

structure of the dual algebra shows that the representations (2.10) [see Appendix, Eq.(A.9)] lift to the structure; there are a homomorphism $\Phi, \Phi' : A \rightarrow U_{< q >, a}(gl_N)$ given by

$$\Phi(X_j^i) = (1/a - 1)q^{ji}Q_i^j, \quad i < j, \quad \Phi(Y_i^j) = 0,$$

$$\Phi(z_k) = \left(\prod_i (q^{ki})^{H_i} \right) a^{H_{k+1} + \dots + H_N}, \quad (5.1)$$

$$\Phi'(Y_i^j) = (a - 1)q^{ji}P_j^i, \quad i < j, \quad \Phi'(X_i^j) = 0,$$

$$\Phi'(z_k) = \left(\prod_i (q^{ki})^{H_i} \right) a^{-H_1 - \dots - H_{k-1}}. \quad (5.2)$$

One can view the universal T-matrix as an element of $U_{< q >, a}(gl_N^*) \otimes U_{< q >, a}(gl_N)$. Then we have

Theorem. The universal R-matrix for $U_{< q >, a}(gl_N)$ is found by applying the mapping $id \otimes \Phi$ to the universal T -matrix (3.6),

$$UR = (id \otimes \Phi)UT = \prod_{\substack{i > j \\ k}} e_a^{\Phi(X_i^j)P_i^j} [\Phi(z_k)]^{H_k}. \quad (5.3)$$

6. Further Generalizations

6.1. STANDARD QUANTUM GROUPS

These are the 1-parameter deformations given by Drinfeld in [1]. Much of their structure is summed up in the quantum double construction; it is basically the same as for twisted $gl(N)$. There are representations defined by (2.10), with kernels I_{\pm} and associated quotients A_{\pm} , and the structure formula (2.9),

$$A = A_- \otimes_{A_0} A_+$$

holds generally, with A_0 the Abelian quotient by $I_+ \cup I_-$. As for twisted $gl(N)$, these representations lift to isomorphisms Φ and Φ' from A_\pm to subalgebras U_\pm of U . The Universal R-matrix is known [19] (hence, so are Φ and Φ'), actually in two forms, $R_+ \in U_- \otimes U_+$ and $R_- \in U_+ \otimes U_-$. The Universal T-matrices for A_\pm are

$$T^- = (\Phi^{-1} \otimes 1)R_-, \quad T^+ = (\Phi'^{-1} \otimes 1)R_-$$

and for A , it is

$$UT = T^- T^+.$$

6.2. NONSTANDARD QUANTUM GROUPS

These include the constructions on $A_n^{(2)}$ and $D_n^{(2)}$ of Jimbo [2], as well as “Esoteric Quantum $gl(N)$ ” and the other deformations of twisted quantum $gl(N)$. It seems that the general formulas that apply to the standard quantum groups have direct generalizations to these cases as well. However, the structure of the quantum double is very different and A_0 is no longer abelian. See [10], second paper.

6.3. ROOTS OF UNITY

The most interesting aspects of quantum groups are those that appear at special values of the parameters; up to this point, in order to postpone the discussion of these phenomena, we have assumed $(\langle q \rangle, a)$ in generic position.

Recall that the exponential form $UT = e^{xp}$ appeared in our first example, in subsection (1.3), as the series $\sum F_k x^k$ with F_k subject to the recursion relation (1.10),

$$F_k F_1 = (k+1)F_{k+1} \quad \Rightarrow \quad F_k = p^k/k!. \quad (6.1)$$

In the evaluation of UT for the quantum groups one encounters modified exponentials,

$$e_a^{xp} = \sum (xp)^n / [n!]_a, \quad (6.2)$$

arising from the solution of the recursion relation

$$F_k F_1 = [k+1]_a F_{k+1} \quad \Rightarrow \quad F_k = p^k / [k]_a. \quad (6.3)$$

See Eq.(3.5). This is the only place that the restriction to generic parameters is relevant. There are no restrictions on the parameters q^{ij} . The limitation to parameters in generic position applies only to the Hecke parameter a , and the (till now) excluded values are those for which there is an integer K such that

$$a^K = 1, \quad a^n \neq 1, \quad n = 1, 2, \dots, K-1.$$

We now suppose that this relation holds; thus a is a primitive root of unity.

Then $[K]_a = 0$, and the general solution of the recursion relation (6.3) is

$$F_{mK+n} = (p'^m / m!) (p^n / [n]_a), \quad p^K = 0,$$

$$m = 0, 1, \dots; \quad n = 0, 1, \dots, K-1.$$

This involves a new generator p' , and the new relation $p^K = 0$. The expressions obtained for the Universal T-matrices are thus modified, when a is a root of unity, by the replacement of all twisted exponentials, according to the rule

$$e_a^{xp} \rightarrow \sum_m (x^K p')^m / m! \sum_{n=0}^{K-1} (xp)^n / [n]_a,$$

with $p^K = 0$.

What is significant is the appearance of a new independent generator p' that replaces p^K . The structure of the dual algebra is drastically changed; it is no longer the Drinfeld quantum group $U_{<q>,a}(\mathcal{G})$. Of course, this latter still exists at roots of unity, but its dual is not the pseudogroup $A(<q>, a)$.

The structure of the dual of $A(<q>, a)$ at $a^K = 1$ can be found by the same methods as in the generic case, but much more quickly by the prescription

$$p' := \lim_{a^K \rightarrow 1} p^K / [K]_a.$$

Thus one obtains for the dual of $A(< q >, a)$ the additional relations

$$[P_i, P_j] = 0, \quad [P'_i, P'_j] = 0,$$

$$[H_k, P_i'^j] = K(\delta_{ki} - \delta_{kj})P_i'^j,$$

$$[P_i, Q'_i] = (a - 1)(q^{12})^{1-H_1-H_2} (Q_1^{K-1} a^{-H_2} - a^{-H_1} Q_1^{K-1}).$$

The last relation is noteworthy; it shows that the generators P'_i **do not** generate an ideal; consequently, the irreducible modules **do not** become nondecomposable at roots of unity.

The dual of $U_{< q >, a}(gl_N)$ can be obtained in the same way. The result is that $(X_i^j)^K$ and $(Y_i^j)^K$ vanish and new independent generators $X_i'^j$ and $Y_i'^j$ appear. In this case $X_i'^j$ and $Y_i'^j$ do generate ideals.

In the simplest case of $U_{q, q'}(gl_2)$ the dual pseudogroup is generated by X, X', ρ_1, ρ_2, Y and Y' , with relations

$$X^K = Y^K = 0, \quad q = e^h, \quad q' = e^{h'},$$

$$[\rho_1, X] = hX, \quad [\rho_1, X'] = KhX',$$

$$[\rho_2, X] = h'X, \quad [\rho_2, X'] = Kh'X', \text{ etc.}$$

The associated Woronowicz matrix; that is, UT evaluated in the 2-dimensional representation of $gl(2)$, is

$$\begin{pmatrix} 1 & 0 \\ X & 1 \end{pmatrix} \begin{pmatrix} e^{\rho_1} & 0 \\ 0 & e^{\rho_2} \end{pmatrix} \begin{pmatrix} 1 & Y \\ 0 & 1 \end{pmatrix},$$

in which X' and Y' do not appear. To obtain the full dual one has to take UT in a representation of quantum $gl(N)$ in which $P^K \neq 0$.

Duality at roots of unity has already been described by Frölich and Kerler [20], for the case of standard quantum $sl(2)$.

Classical q -functions are always investigated in the domain $|q| < 1$. Indeed, many of these functions, including the twisted exponential, cease to exist as q tends to a root of unity. We have seen that the twisted exponential, within the algebraic context, does have a natural generalization to roots of unity: but that this generalization is highly nontrivial and even mysterious in a purely analytical context. It is proposed to take up the study of q -functions on a broader basis, with a view to finding natural extensions to roots of unity. Perhaps even the Rogers-Ramanujan identities can be generalized to $q^K = 1$.

7. Quantum $gl(N)$ and Quantum $sl(N)$.

The (Drinfeld) quantum group $U_{<q>,a}(sl_N)$ is the subquotient of $U_{<q>,a}(gl_N)$ defined by the ideal generated by the element

$$\sum_{i=1}^N H_i \equiv \mathcal{H}.$$

The universal T- and R-matrices of quantum $gl(N)$ reduce to those of quantum $sl(N)$ under the projection that annuls \mathcal{H} . We shall calculate this reduced R-matrix, in the fundamental representation of $sl(N)$.

Restricting the Universal R-matrix (5.3) to any faithful N -dimensional representation gives the formula

$$R = \left[1 + \sum_{i>j} P_i^j \Phi(X_i^j) \right] \sum_{i,j} (\tilde{q}^{ij})^{H_i \oplus H_j}, \quad (7.1)$$

with $\tilde{q}^{ij} = aq^{ij}$ for $i < j$ and $\tilde{q}^{ij} = q^{ij}$ for $i \geq j$. With $\mathcal{H} = 0$, in the N -dimensional representation of $sl(N)$ we have $H_i = M_i^i - 1/N$. Substituting this into (7.1) we obtain

$$R = (1 - a) \sum_{i<j} (\kappa_i / \kappa_j M_j^i \otimes M_i^j + \sum_{i,j} \hat{q}^{ij} M_i^i \otimes M_j^j), \quad (7.2)$$

with

$$\hat{q}^{ij} = (\kappa_i/\kappa_j)q^{ij}, \quad \kappa_i = \left(a^i \prod_k q^{ki} \right)^{1/N}. \quad (7.3)$$

This R-matrix for $sl(N)$ differs from that of $gl(N)$ in two particulars. (1) q^{ij} is replaced by \hat{q}^{ij} . These new parameters are not independent:

$$\prod_i \hat{q}^{ij} a^j = a^{(N+1)/2}. \quad (7.4)$$

If the original parameters satisfy

$$\prod_i q^{ij} a^j = a^{(N+1)/2}, \quad (7.5)$$

then $\hat{q}^{ij} = q^{ij}$. Therefore, reducing from $gl(N)$ to $sl(N)$ as we have done, from arbitrary initial parameters, gives the same result as restricting the parameters to satisfy (7.5). (2) The factor κ_i/κ_j can be removed by the isomorphism $M_i^j \rightarrow (\kappa_i/\kappa_j)M_i^j$.

The restriction (7.5) was found by Schirmacher [12] from the requirement that the quantum determinant be unity.

The deformations of $A(< q >, a)$ have recently been calculated [15]. This algebra is rigid for essential, first order deformations except for very special values of the parameters. There are several series of deformations of twisted quantum $gl(N)$, here we illustrate the simplest one, in the case of $gl(3)$.

The parameters are q^{12}, q^{23}, q^{13} and a . The deformation that consists of adding the following piece

$$\delta R = \epsilon (q^{13} M_1^2 \otimes M_3^2 - M_3^2 \otimes M_1^2)$$

to (1.18), is exact and essential. It exists if and only if the parameters satisfy

$$q^{12} = q^{23} \quad \text{and} \quad q^{13} = (q^{12})^2.$$

The projection to $sl(3)$ fixes the value of a :

$$q^{12} = q^{23} := q, \quad q^{13} = q^2. \quad a = q^{-3}.$$

That is why, for $sl(3)$, unlike $gl(3)$, the term “roots of unity” applies to the values of the q 's.

This esoteric form of quantum $sl(3)$ is not included in the list produced by Jimbo [2], but the classical limit is in the classification of Belavin and Drinfeld [21].

Appendix

The R -matrix for twisted, quantum $gl(N)$, in the fundamental representation, was given in (1.18),

$$R = \sum_i M_i^i \otimes M_i^i + \sum_{i < j} (q^{ji} M_j^j \otimes M_i^i + a q^{ij} M_i^i \otimes M_j^j + (1 - a) M_j^j \otimes M_i^i),$$

The inverse matrix is given by the same formula, with the parameters q^{kl} and a replaced by their inverses.

The relations (2.4) of $A(< q >, a)$, written out in full detail, are

$$\begin{aligned} z_i^a z_i^b &= q^{ab} z_i^b z_i^a, \\ z_i^a z_j^a &= (a q^{ij})^{-1} z_j^a z_i^a, \quad i < j, \\ z_i^a z_j^b &= (a q^{ab} / q^{ij}) z_j^b z_i^a, \quad i > j, \quad a < b \\ q^{ij} z_i^a z_j^b - q^{ab} z_j^b z_i^a &= (a - 1) z_j^a z_i^b, \quad i > j, \quad a > b. \end{aligned} \tag{A.1}$$

The quotient algebras A_+, A_- satisfy relations that one gets from these by annulling the generators in I_+, I_- ,

$$A_- : z_i^j \rightarrow \begin{cases} \tilde{X}_i^j, & j \leq i \\ 0, & i < j \end{cases} ; \quad A_+ : z_i^j \rightarrow \begin{cases} \tilde{Y}_i^j, & i \leq j \\ 0, & j < i \end{cases} \tag{A.2}$$

Finally, one gets from (2.12) the following relations for A_- ,

$$\begin{aligned}
x_i x_j &= x_j x_i, \\
x_k X_i^j &= \begin{cases} q^{ik} q^{kj} X_i^j x_k, & k < j < i \text{ or } j < i \leq k, \\ (1/a) q^{ik} q^{kj} X_i^j x_k, & j \leq k < i \end{cases} \quad (A.3)
\end{aligned}$$

These become relations of A after substitution of z_i for x_i . Likewise, the following relations for A^+ become relations of A if y_i is replaced by z_i ,

$$\begin{aligned}
y_i y_j &= y_j y_i, \\
y_k Y_i^j &= \begin{cases} q^{ik} q^{kj} Y_i^j y_k, & k \leq i < j \text{ or } i < j < k, \\ a q^{ik} q^{kj} Y_i^j y_k, & i < k \leq j. \end{cases} \quad (A.3')
\end{aligned}$$

The coefficients in (2.15-19) are

$$\begin{aligned}
k_{ij} &= q^{i+1,j} q^{ji} / q^{i+1,j+1} q^{j+1,i}, \\
k_i &= q^{i+1,i-1} / q^{i+1,i} q^{i,i-1}. \quad (A.4)
\end{aligned}$$

We need some formulas for the coproduct of A_- . From (2.12) and

$$\Delta(\tilde{X}_i^j) = \sum_k \tilde{X}_i^k \otimes \tilde{X}_k^j \quad (A.5)$$

one gets

$$\Delta(x_i) = x_i \otimes x_i, \quad (A.6)$$

$$\Delta(X_i^j) = \sum_k X_i^k (x_k/x_j) \otimes X_k^j = X_i^j \otimes 1 + (x_i/x_j) \otimes X_i^j + \dots \quad (A.7)$$

Only the first two terms are relevant for our calculation of T^- ; neglecting the rest we have

$$\Delta(X) = A + B, \quad A = X_i^j \otimes 1, \quad B = (x_i/x_j) \otimes X_i^j. \quad (A.8)$$

One has $BA = aBA$ from (A.3) and thus $(A+B)^n = A^n + A^{n-1}B(1+a+\dots+a^{n-1})$ and finally the result

$$\Delta(X_i^j)^n = (X_i^j)^n \otimes 1 + [n]_a (X_i^j)^{n-1} (x_i/x_j) \otimes X_i^j + \dots,$$

that was used in (3.4).

For the evaluation of $[P_i, Q_i]$ we need to know some terms in $\Delta(z_i)$,

$$\begin{aligned} \Delta(z_k) &= (z_k z'_k)^{H_k} \left(1 + (q^{k,k+1}/a) Y_k X'_k - q^{k-1,k} X'_{k-1} Y_{k-1} \dots \right) \\ (\Delta(z_k))^{H_k} &= (z z')^{H_k} \left(1 + [H_k]_a (q^{k,k+1}/a) Y_k X'_k - [H_k]_{1/a} q^{k-1,k} Y_{k-1} X'_{k,k-1} + \dots \right). \end{aligned}$$

The relation (4.6) now follows easily in the same way.

The representations π and π' are given by (2.10) and (1.18), and more explicitly by vskip-3mm

$$\begin{aligned} \pi(z_j^i) &= \begin{cases} (1-a)M_i^j, & i < j, \\ 0, & i > j, \end{cases} \quad \pi(z_k^j)_i^j = \delta_i^j q^{ki} a^{(i>k)}, \\ \pi'(z_i^j) &= \begin{cases} (1-1/a)M_j^i, & i < j, \\ 0, & i > j, \end{cases} \quad \pi'(z_k^j)_i^j = \delta_i^j q^{ki} a^{-(i<k)}, \end{aligned} \quad (A.9)$$

where $(i < k) = 1$ if $i < k$ and zero otherwise.

Acknowledgments

This work was supported in part by the National Science Foundation.

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